Last time:

* AC circuits: Has a source $V(t) = V_0 \sin \omega t$

current is given:

$$I(t) = I_0 \sin(\omega t - \phi) \quad \text{where} \quad I_0 \equiv \text{amplitude}, \quad \phi \equiv \text{phase}$$

* For AC circuits with only one element:

	Resistance/reactance	Current amplitude	Phase ϕ
R	R	$I_{R_0} = \frac{V_0}{R}$	0
L	$X_L = \omega L$	$I_{L_0} = \frac{V_0}{X_L}$	$\pi/2$ current lags voltage
C	$X_C = \frac{1}{\omega C}$	$I_{C_0} = \frac{V_0}{X_C}$	$-\pi/2$ current leads voltage

* For an RLC circuits

	Impedance Z	Current amplitude	Phase angle
R L C (series)	$\left[R^2 + (X_L - X_C)^2\right]^{1/2}$	$I_0 = \frac{V_0}{\left[R^2 + (X_L - X_C)^2\right]^{1/2}}$	$\tan^{-1}\left[\frac{X_L - X_C}{R}\right]$
R L C (parallel)	$\frac{1}{Z} = \left[\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2\right]^{1/2}$	$I_0 = V_0\left[\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2\right]^{1/2}$	$\tan^{-1}\left[R\left(\omega C - \frac{1}{\omega L}\right)\right]$

* The root mean square voltage (rms)

$$V_{rms} = \frac{V_0}{\sqrt{2}} \quad ; \quad I_{rms} = \frac{I_0}{\sqrt{2}}$$

Today: Maxwell's equations

(i) $\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0}\rho$ Gauss's law

(ii) $\nabla \cdot \underline{B} = 0$ "No name"

(iii) $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$ Faraday's law $\leftarrow$

(iv) $\nabla \times \underline{B} = \mu_0 \underline{J}$ Ampère's law $\leftarrow$

What is wrong here? There is a mathematical inconsistency.

$\nabla \cdot (\nabla \times \underline{A}) = 0$ for any $\underline{A}$

What happens when you calculate the div of eq $\leftarrow$?

Let's check for eq. (iii)

$$\nabla \cdot (\nabla \times \underline{E}) = \nabla \cdot \left(\frac{-\partial \underline{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\overset{0}{\cancel{\nabla \cdot \underline{B}}}\right) = 0 \quad \text{eq (iii)} \checkmark$$

eq (ii)

Now we calculate $\nabla \cdot (\nabla \times \underline{B})$ eq iv:

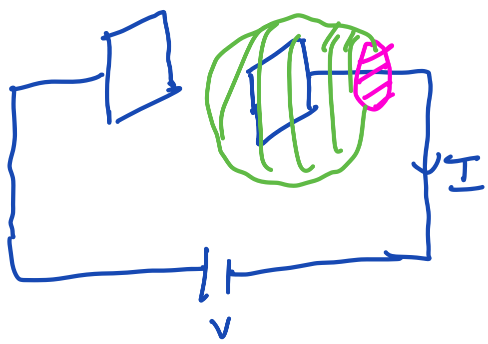
$$\nabla \cdot (\nabla \times \underline{B}) = \nabla \cdot (\mu_0 \underline{J}) = \mu_0 (\nabla \cdot \underline{J})$$

For steady currents
$\nabla \cdot \underline{J} = 0$

beyond magnetostatics $\underline{J}$ is not steady so $\nabla \cdot \underline{J} = -\frac{\partial \rho}{\partial t} \neq 0$

There is a problem with Ampère's law.

Why does Ampère law no longer hold for non steady currents?



We look at the situation where we are charging up the capacitor.

In integral form Ampère's law:

$$\oint \underline{B} \cdot d\underline{l} = \mu_0 I_{enc}$$

Stokes

$$\oint \underline{B} \cdot d\underline{l} = \int_S \nabla \times (\underline{B} \cdot d\underline{a})$$

How do we determine I_{enc}? I_{enc} is the total current passing through the loop. For the surface  enclosed by the loop I is flowing and $\nabla \times \underline{B} = \mu_0 \underline{J}$.

What happens if we consider  no I!

We never had this problem in magnetostatics, this arises only when charges pile up somewhere. There must be a missing term that doesn't depend on $\underline{J}$:

$$\nabla \times \underline{B} = \mu_0 \underline{J} + (?) \swarrow$$

For non-steady currents "current enclosed by the loop" is ill defined → depends on surface chosen.
This a mathematical inconsistency with the theory.

How was Ampère's law fixed?

What is (?) in the eq. for the curl of $\underline{B}$. Let $(?) \equiv \underline{F}$

$$\nabla \times \underline{B} = \mu_0 \underline{J} + \underline{F} \qquad \text{Find } \underline{F} \text{ such that } \nabla \cdot (\nabla \times \underline{B}) = 0$$

$$\nabla \cdot (\nabla \times \underline{B}) = \nabla \cdot (\mu_0 \underline{J} + \underline{F}) = 0$$

$$\Rightarrow \mu_0 \nabla \cdot \underline{J} + \nabla \cdot \underline{F} = 0 \Rightarrow \mu_0 \nabla \cdot \underline{J} = -\nabla \cdot \underline{F}$$

$$\Rightarrow \nabla \cdot \underline{F} = \mu_0 \left(\frac{\partial \rho}{\partial t} \right) \quad (*)$$

This term looks similar to Gauss's law

$$\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0} \rho$$

$$\uparrow$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0$$

$$\Rightarrow \nabla \cdot \underline{J} = -\frac{\partial \rho}{\partial t}$$

If we take the time derivative of Gauss's law:

$$\frac{\partial}{\partial t} (\nabla \cdot \underline{E}) = \frac{\partial}{\partial t} \left(\frac{1}{\varepsilon_0} \rho \right) = \frac{1}{\varepsilon_0} \frac{\partial \rho}{\partial t}$$

$$\Rightarrow \nabla \cdot \frac{\partial \underline{E}}{\partial t} = \frac{1}{\varepsilon_0} \underbrace{\frac{\partial \rho}{\partial t}}_{\frac{\nabla \cdot \underline{F}}{\mu_0}} = \frac{1}{\varepsilon_0 \mu_0} \nabla \cdot \underline{F}$$

$$\therefore \nabla \cdot \frac{\partial \underline{E}}{\partial t} = \frac{1}{\varepsilon_0 \mu_0} \nabla \cdot \underline{F} \Rightarrow \underline{F} = \varepsilon_0 \mu_0 \left(\frac{\partial \underline{E}}{\partial t} \right)$$

We can write the generalized Ampère's law:

$$\nabla \times \underline{B} = \mu_0 \underline{J} + \varepsilon_0 \mu_0 \left(\frac{\partial \underline{E}}{\partial t} \right)$$

- For magnetostatics this doesn't change anything.
- For ordinary experiments the second term doesn't matter too much. But it's really important when discussing the propagation of EM waves.
- This makes the equations symmetric

$$\boxed{\text{A changing electric field induces a magnetic field}}$$

- Maxwell called this second term the displacement current.

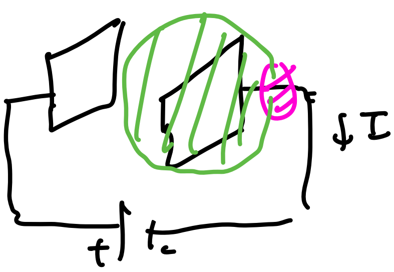
$$\boxed{\underline{J}_d \equiv \varepsilon_0 \frac{\partial \underline{E}}{\partial t}}$$

Using this notation Ampère's law becomes

$$\nabla \times \underline{B} = \mu_0 \underline{J} + \mu_0 \underline{J}_D = \mu_0 (\underline{J} + \underline{J}_d)$$

But $\underline{J}_d$ is not an actual current, physically it does NOT describe charge flowing through some region.

With this fix we can look again at the charging capacitor



For a ∥ plate capacitor the approx electric field:

$$E = \frac{1}{\varepsilon_0} \sigma = \frac{1}{\varepsilon_0} \frac{Q}{A}$$ where Q is charge, A area

so between plates:

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 A} \frac{dQ}{dt} = \frac{1}{\varepsilon_0 A} I$$

Now using the generalized Ampère's law in integral form:

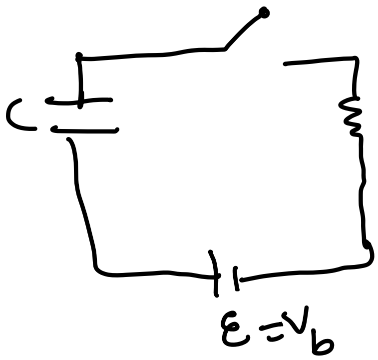
$$\oint \underline{B} \cdot d\underline{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \int \left(\frac{\partial \underline{E}}{\partial t} \right) \cdot d\underline{a}$$

For $E = 0$ and $I_{enc} = I$

For $J_{enc} = 0$ but $\int \left(\frac{\partial E}{\partial t}\right) \cdot da = \frac{I}{\varepsilon_0}$ ✓ using eq.

we fixed the inconsistency.

The importance of displacement currents:



We said some current flows through all circuit elements.

→ Is there current flow in the capacitor?

We can think of displacement currents continuing the "real" current across the capacitor ensuring the validity of Kirchhoff's rules.

Example: calculate $\underline{B}$ inside the plates of the circuit.

What is the magnetic field inside the plates of the capacitor? Assume you have cylindrical plates of radius a.

① First we will calculate $\underline{E}$ between plates.

$$\underline{E} = \frac{\sigma}{\varepsilon_0} = \frac{1}{\varepsilon_0}\frac{Q(t)}{A} = \frac{1}{\varepsilon_0}\frac{Q(t)}{\pi a^2}$$

② From this we can calculate $\underline{J}_d$

$$\underline{J}_d = \varepsilon_0 \frac{\partial \underline{E}}{\partial t} = \frac{\varepsilon_0}{\varepsilon_0 \pi a^2}\frac{\partial Q}{\partial t} = \frac{I(t)}{\pi a^2}$$

③ For a capacitor

$$I(t) = \frac{V_b}{R} e^{-t/RC} \Rightarrow \underline{J}_d = \frac{1}{\pi a^2}\frac{V_b}{R} e^{-t/RC}$$

④ Now the magnetic field

$$\underbrace{\oint \underline{B} \cdot d\underline{l}}_{2\pi r} = \mu_0 \cancelto{0}{I_{enc}} + \mu_0 \int \underline{J}_d \cdot d\underline{a} = \frac{\mu_0}{\pi a^2}\frac{V_b}{R} e^{-t/RC} \underbrace{(\pi r^2)}_{\int da}$$

$$\Rightarrow B(r) = \frac{\mu_0 r V_b\, e^{-t/RC}}{2\pi a^2 R}$$

We have now arrived at Maxwell's equations:

(i) $\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0}\rho$ (iii) $\nabla \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0$

(ii) $\nabla \cdot \underline{B} = 0$ (iv) $\nabla \times \underline{B} - \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{J}$

All of the physics of this course are contained in these equations + Lorentz Force law $\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$

* This emphasizes sources of $\underline{E}$ and $\underline{B}$ are charges and currents.

* The sources for $\frac{\partial \underline{E}}{\partial t}$ and $\frac{\partial \underline{B}}{\partial t}$ are also charges and currents

In integral form:

(i) $\oint \underline{E} \cdot d\underline{a} = \frac{Q}{\varepsilon_0}$ (iii) $\oint \underline{E} \cdot d\underline{l} = -\frac{d\Phi_B}{dt}$

(ii) $\oint \underline{B} \cdot d\underline{a} = 0$ (iv) $\oint \underline{B} \cdot d\underline{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$

What is the meaning of these equations:

(i) Electric flux through closed surface is proportional to charges enclosed

(ii) The total magnetic flux through a closed surface is zero
$\Rightarrow$ non-existence of magnetic monopoles

(iii) changing flux of magnetic field produces an electric field.

(iv) Electric current and changing electric flux produces a magnetic field.

Maxwell's equations in vacuum

In vacuum $\Rightarrow \rho = 0$ and $\underline{J} = 0$ Maxwell's eq become:

(i) $\nabla \cdot \underline{E} = 0$ (iii) $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$

(ii) $\nabla \cdot \underline{B} = 0$ (iv) $\nabla \times \underline{B} = \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$

except for a minus sign the eq for $\underline{E}$ and $\underline{B}$ are symmetric.

We will analyze the solutions to these equations.

To solve them we need to uncouple them. We can do so by taking the curl of eq (iii):

$$\nabla \times (\nabla \times \underline{E}) = \nabla(\cancel{\nabla \cdot \underline{E}})^{0} - \nabla^2 \underline{E} = -\nabla^2 \underline{E} \quad \text{for vacumm}$$

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Taking the curl of the rhs of the eq:

$$\nabla \times \left(\frac{-\partial B}{\partial t}\right) = -\frac{d}{dt}(\nabla \times \underline{B}) = -\frac{d}{dt}\left(\mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}\right) \text{ using eq IV.}$$

Equating both sides we get

$$\boxed{\nabla^2 \underline{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}}$$

we can repeat this for eq (IV):

$$\nabla \times (\nabla \times \underline{B}) = \nabla(\cancel{\nabla \cdot \underline{B}})^{0} - \nabla^2 \underline{B} = -\nabla^2 \underline{B}$$

And the rhs of the eq:

$$\nabla \times \left(\mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}\right) = \mu_0 \varepsilon_0 \frac{d}{dt}(\nabla \times \underline{E}) = \mu_0 \varepsilon_0 \frac{d}{dt}\left(\frac{-\partial \underline{B}}{\partial t}\right)$$

$$= -\mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2}$$

So equating both sides:

$$\boxed{\nabla^2 \underline{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{B}}{\partial t^2}}$$

These equations are similar to the wave equation

$$\boxed{\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}}$$

The wave equation

where

$f = f(x \pm vt)$ is a solution of the equation.

where v is the velocity of the wave.

$\underline{B}$ and $\underline{E}$ satisfy the wave equation

$$\boxed{v = \frac{1}{(\varepsilon_0 \mu_0)^{1/2}} = 3\times10^8\, m/s \equiv c \quad \text{speed of light}}$$

Maxwell's equations in matter.

It's convenient to write things in terms of the auxiliary fields $\underline{D}$ and $\underline{H}$.

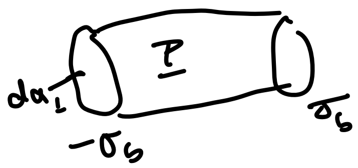
In the static case, electric polarization ($\underline{P}$) produces a bound charge density:

$$\rho_b = -\nabla\cdot\underline{P}$$

Similarly magnetization produces a bound current

$$\underline{J}_b = \nabla\times\underline{M}$$

To generalize this we need to consider that any change the electric polarization will involve a flow of bound charge we will call this $\underline{J}_p$



$$dI = \frac{\partial\sigma_b}{\partial t}\, da_\perp = \frac{\partial P}{\partial t}\, da_\perp = \frac{\partial\underline{P}}{\partial t} \equiv \underline{J}_p$$

polarization current.

$\underline{J}_p$ is the result of moving charges as the polarization changes.

This fulfills conservation of charge

$$\nabla\cdot\underline{J}_p \underset{\substack{\uparrow \\ \text{definition} \\ \text{of } \underline{P}}}{=} \nabla\cdot\frac{\partial\underline{P}}{\partial t} = \frac{\partial}{\partial t}\left(\nabla\cdot\underline{P}\right) = -\frac{\partial\rho_b}{\partial t}$$

If magnetization changes we don't have an accumulation of charge or a current.

$$\underline{J}_b = \nabla\times\underline{M}$$ responds to changes in magnetization without accumulation of charge.

The possible source for charge:

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \underline{P}$$

We have an additional current density due to changing $\underline{P}$:

$$\underline{J} = \underline{J}_f + \underline{J}_b + \underline{J}_p = \underline{J}_f + \nabla \times \underline{M} + \frac{\partial \underline{P}}{\partial t}$$

With these sources we can rewrite Maxwell's eq:

$$\nabla \cdot \underline{E} = \frac{1}{\varepsilon_0} \left(\rho_f - \nabla \cdot \underline{P} \right) \Rightarrow \nabla \cdot \underline{D} = \rho_f$$

remembering $\underline{D} = \varepsilon_0 \underline{E} + \underline{P}$

And Ampère's law w/ Maxwell's term becomes:

$$\nabla \times \underline{B} = \mu_0 \left(\underline{J}_f + \nabla \times \underline{M} + \frac{\partial \underline{P}}{\partial t} \right) + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$$

$$\Rightarrow \nabla \times \underline{H} = \underline{J}_f + \frac{\partial \underline{D}}{\partial t}$$

remembering

$$\underline{H} = \frac{1}{\mu_0} \underline{B} - \underline{M}$$

$$\underline{D} = \varepsilon_0 \underline{E} + \underline{P}$$

In matter:

(i) $\nabla \cdot \underline{D} = \rho_f$

(ii) $\nabla \cdot \underline{B} = 0$

(iii) $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$

(iv) $\nabla \times \underline{H} = \underline{J}_f + \frac{\partial \underline{D}}{\partial t}$

with constitutive relationships. For linear media

$$\underline{P} = \varepsilon_0 \chi_e \underline{E} \quad \text{and} \quad \underline{M} = \chi_m \underline{H}$$

so

$$\underline{D} = \varepsilon \underline{E} \quad \text{and} \quad \underline{H} = \frac{1}{\mu} \underline{B}$$

The equations in integral form:

(i) $\oint_S \underline{D} \cdot d\underline{a} = Q_{f\,enc}$

(ii) $\oint_S \underline{B} \cdot d\underline{a} = 0$

(iii) $\oint \underline{E} \cdot d\underline{l} = -\frac{d}{dt} \int \underline{B} \cdot d\underline{a}$

(iv) $\oint_S \underline{H} \cdot d\underline{l} = I_{f\,enc} + \frac{d}{dt} \int \underline{D} \cdot d\underline{a}$